Essence of Linear Algebra
Some cool intuitions

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Disclaimer

All the credits for the content goes to the respective authors listed in the Appendix: Further Learning.
The Storyline

The Geometry of Linear Equations
  Vectors and Basis vectors
  Linear combinations and Span

The box game: Matrices
  Elimination and Multiplication, A=LU

Transforming your LIFE Leenearly
  Cool Video, The Determinant

Space Tour
  Column space, Null space, Inverses

Celebrity: The Rank
  Solution concept

Some things of Eigen
  Eigen values, Eigen vectors, Change of Basis
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Vectors

What even are they?

- A line with a arrowhead? OR

▶ A line with a arrowhead? OR
Vectors

What even are they?

- A line with a arrowhead?
  OR
- A set of numbers arranged vertically?

OR

\[
\begin{bmatrix}
1 \\
2 \\
3
\end{bmatrix}
\]
Vectors

What even are they?

- A line with a arrowhead? OR
- A set of numbers arranged vertically?

\[
\begin{bmatrix}
1 \\
2 \\
3 \\
\end{bmatrix}
\]

OR

- An abstract \( \vec{v} \)
Vectors

Abstract view

\[ \vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w} \]

\[ \vec{v} + \vec{w} = \vec{w} + \vec{v} \]

There is a zero vector \( \vec{0} \) such that \( \vec{0} + \vec{v} = \vec{v} \) for all \( \vec{v} \). For every vector \( \vec{v} \) there is a vector \(-\vec{v}\) so that \( \vec{v} + (-\vec{v}) = \vec{0} \).

\[ a(b\vec{v}) = (ab)\vec{v} \quad 1\vec{v} = \vec{v} \]

\[ a(\vec{v} + \vec{w}) = a\vec{v} + a\vec{w} \]

\[ (a + b)\vec{v} = a\vec{v} + b\vec{v} \]

---

\(^1\)Abstract vector spaces — Essence of Linear Algebra, Chapter 11
Basis Vectors

These are the vectors which can define the entire coordinate space.

▶ Do you recognize this special vector?

\[
\begin{bmatrix}
\vec{i} \\
\vec{j} \\
\vec{k}
\end{bmatrix}
\]
Basis Vectors

These are the vectors which can define the entire coordinate space.

▶ Do you recognize this special vector?

\[
\begin{bmatrix}
\vec{i} \\
\vec{j} \\
\vec{k}
\end{bmatrix}
\]

▶ Also, have you seen this?

\[
\begin{bmatrix}
1 & 1 \\
0 & 1
\end{bmatrix}
\]

This is a special matrix called Shear matrix.
Basis vectors

Example

The image\(^2\) below shows the basis vectors of a Shear matrix.

\[
\begin{bmatrix}
1 & 1 \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
x \\
y
\end{bmatrix}
\]

\(^2\)Essence of Linear Algebra, Chapter 3
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Linear combinations

Additivitiy

\[
\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 6 \\ 3 \end{bmatrix} = \begin{bmatrix} 7 \\ 5 \end{bmatrix} \quad \vec{u} + \vec{v} = \vec{w}
\]

Scaling

\[
2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix} \quad 2\vec{v} = (2\vec{v})
\]

Hybrid

\[
2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 3 \begin{bmatrix} 6 \\ 3 \end{bmatrix} = \begin{bmatrix} 20 \\ 13 \end{bmatrix} \quad a\vec{u} + b\vec{v} = \vec{w}
\]
Span

Example

The image\(^3\) below shows how two vectors can span the 2D space.

\(^3\)Essence of Linear Algebra, Chapter 2
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Elimination

Gauss-Jordan Elimination
A method of solving a linear system of equations. This is done by transforming the system’s augmented matrix into reduced row-echelon form (rref) by means of row operations.

Types of row Operations:
Type 1: Swap the positions of two rows.
Type 2: Multiply a row by a nonzero scalar.
Type 3: Add to one row a scalar multiple of another.

Example

\[
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 2 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

\[ \text{RREF} = \]
Elimination using Multiplication

Example

\[ \begin{align*}
    x + 2y + z &= 2 \\
    3x + 8y + z &= 12 \\
    4y + z &= 2
\end{align*} \]

\[
\begin{bmatrix}
    1 & 0 & 0 \\
    0 & 1 & 0 \\
    0 & -2 & 1
\end{bmatrix}
\begin{bmatrix}
    1 & 0 & 0 \\
    -3 & 1 & 0 \\
    0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
    1 & 2 & 1 \\
    3 & 8 & 1 \\
    0 & 4 & 1
\end{bmatrix}
= \begin{bmatrix}
    1 & 2 & 1 \\
    0 & 2 & -2 \\
    0 & 0 & 5
\end{bmatrix}
\]

\[ E_{32}E_{31}E_{21}A = U \]
Example

\[
\begin{bmatrix}
1 & 2 & 1 \\
3 & 8 & 1 \\
0 & 4 & 1
\end{bmatrix}
= 
\begin{bmatrix}
1 & 0 & 0 \\
3 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 2 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 2 & 1 \\
0 & 2 & -2 \\
0 & 0 & 5
\end{bmatrix}
\]

\[EA = U\]

\[A = (E_{32}E_{31}E_{21})^{-1}U\]

\[A = E_{21}^{-1}E_{31}^{-1}E_{32}^{-1}U\]

\[A = LU\]

So, \[A = E^{-1}U\]. Therefore, \[L = E^{-1}\]
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Cool Video

Definition Cool
The phrase ”cool” is very relaxed, never goes out of style, and people will never laugh at you for using it.

Definition Video
Make a video recording of (something broadcast on television). YouTube nowadays.

Theorem
Cool + Video = JUST START PLAYING THE VIDEO!

Example
Here You Go: HUGO
Cool Video

Shreedhar Kodate

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Summary
The Determinant

The determinant of a matrix $A$ is denoted $\det(A)$ or $\det A$. It can be viewed as the scaling factor of the transformation described by the matrix.\(^4\)

The formula:

$$\det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

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- The determinant can tell us whether or not a given transformation associated with that matrix squishes everything into a smaller dimension.

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The formula:

\[
\det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc
\]

- The determinant can tell us whether or not a given transformation associated with that matrix squishes everything into a smaller dimension.
- Also, if the value of determinant is negative then the transformation is equivalent to inverting the orientation of space.

The Determinant

Example

The image\(^5\) below shows the significance of calculating the Determinant of a matrix.

\[
\begin{vmatrix}
3 & 2 \\
0 & 2 \\
\end{vmatrix} = 6
\]

\(^5\) The Determinant — Essence of Linear Algebra, Chapter 5
The Determinant

Example

The image\(^6\) below shows how to calculate the Determinant of a 2D matrix.

\[ \det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = (a + b)(c + d) - ac - bd - 2bc = ad - bc \]

---

\(^6\)The Determinant — Essence of Linear Algebra, Chapter 5
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Column space and Null space

Column space: \( C(A) \)

The column space of a matrix \( A \) is the vector space generated by all the linear combinations of the column vectors.

Null space: \( N(A) \)

The set of all vectors \( \vec{v} \) such that

\[
A \vec{v} = \vec{0}
\]

Example

\[
\begin{bmatrix}
1 & 1 & 2 \\
2 & 1 & 3 \\
3 & 1 & 4 \\
4 & 1 & 5 \\
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
z \\
\end{bmatrix} =
\begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
\end{bmatrix}
\Rightarrow N(A) = c
\begin{bmatrix}
1 \\
1 \\
-1
\end{bmatrix}
\]
Inverse of a matrix $A$

**Inverse of $A = A^{-1}$**

In terms of transformation, the matrix that undoes all the transformations made by matrix $A$ is called as inverse of matrix $A$ ($A^{-1}$).

$$A^{-1}A = AA^{-1} = I$$

$$Ax = b \Rightarrow A^{-1}Ax = A^{-1}b \Rightarrow x = A^{-1}b$$

**Example**

$$A = \begin{bmatrix} 7 & 2 & 1 \\ 0 & 3 & 1 \\ -3 & 4 & 2 \end{bmatrix} \quad b = \begin{bmatrix} 21 \\ 5 \\ -1 \end{bmatrix} \quad x = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$x = A^{-1}b \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 & 8 & -5 \\ 3 & -11 & 7 \\ 9 & -34 & 21 \end{bmatrix} \begin{bmatrix} 21 \\ 5 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix}$$
Example

The image below shows how two vectors are linked to each other via Inverse transformation.

\[ A\mathbf{x} = \mathbf{v} \]
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Find Mr. Rank

What is the rank of the following matrix?

Be Quick!\(^8\)

\[
\begin{bmatrix}
12 & 15 & 14 & 19 & 13 & 21 & 05 & 07 & 41 & 51 \\
22 & 26 & 26 & 32 & 27 & 36 & 21 & 24 & 59 & 70 \\
10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 \\
24 & 30 & 28 & 38 & 26 & 42 & 10 & 14 & 82 & 102 \\
30 & 33 & 36 & 39 & 42 & 45 & 48 & 51 & 54 & 57 \\
15 & 16.5 & 18 & 19.5 & 21 & 22.5 & 24 & 25.5 & 27 & 28.5
\end{bmatrix}
\]

\(^8\)You’ll be given choc!
# The Rank

## Solution concept provided by The Rank

The Rank tells you everything about the number of solutions to a given system of linear equations.\(^9\)

<table>
<thead>
<tr>
<th>Matrix A with dimensions m x n, rank r and rref(A) = R</th>
<th>r = m = n</th>
<th>r = n &lt; m</th>
<th>r = m &lt; n</th>
<th>r &lt; m, r &lt; n</th>
</tr>
</thead>
</table>
| R = I | R = \[
\begin{bmatrix}
I
\end{bmatrix}
\] | R = \[
\begin{bmatrix}
I \\
F
\end{bmatrix}
\] | R = \[
\begin{bmatrix}
I \\
F
\end{bmatrix}
\] |
| 1 | 0 or 1 | 1 or ∞ | 0 or ∞ |

\(^9\)Gilbert Strang, Linear Algebra lecture 8
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Some things of Eigen

Eigen values and vectors

An Eigen vector of a linear transformation is a non-zero vector whose direction does not change when that linear transformation is applied to it. More formally, if T is a linear transformation from a vector space \( V \) and \( \vec{v} \) is a vector in \( V \) that is not the zero vector, then \( \vec{v} \) is an eigenvector of \( T \) if \( T(\vec{v}) \) is a scalar multiple of \( \vec{v} \). This condition can be written as the equation

\[
T(\vec{v}) = \lambda \vec{v}
\]

where \( \lambda \) is a scalar known as the eigenvalue, characteristic value or root associated with the eigenvector \( \vec{v} \).
Eigen values and vectors

Example

The image\textsuperscript{10} below shows how the eigenvector does not change its direction after applying a linear transformation.

\textsuperscript{10}Eigenvectors and eigenvalues, Essence of linear algebra, chapter 10
Eigen values and vectors

Example

The image\(^\text{11}\) below shows how the eigenvector gets scaled after applying a linear transformation.

\(^{11}\)Eigenvectors and eigenvalues, Essence of linear algebra, chapter 10
Change of Basis

New coordinates to Old coordinates

\[
\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = (-1) \begin{bmatrix} 2 \\ 1 \end{bmatrix} + (2) \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -4 \\ 1 \end{bmatrix}
\]

Old coordinates to New coordinates

\[
\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1/3 & 1/3 \\ -1/3 & 2/3 \end{bmatrix} \Rightarrow \begin{bmatrix} 1/3 & 1/3 \\ -1/3 & 2/3 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 5/3 \\ 1/3 \end{bmatrix}
\]
Change of Basis

Transformation

Let \( \vec{v} \) be a vector in the New coordinates i.e. change of Basis vectors, \( A \) be the matrix representing the transformation: Change of Basis, and \( M \) be the transformation in Old coordinate system and \( T \) be the final transformation in the New coordinate system.

\[
A^{-1}MA\vec{v} = T\vec{v}
\]

Example

\[
\begin{bmatrix}
2 & -1 \\
1 & 1
\end{bmatrix}^{-1}
\begin{bmatrix}
0 & -1 \\
1 & 0
\end{bmatrix}
\begin{bmatrix}
2 & -1 \\
1 & 1
\end{bmatrix}
\vec{v} =
\begin{bmatrix}
1/3 & -2/3 \\
5/3 & -1/3
\end{bmatrix}
\vec{v}
\]
Summary

How do you possibly hope to *summarize the whole talk*?

Like the OLD MAN said: **TOGETHER!**
For Further Learning

- Gilbert Strang
  [https://tinyurl.com/gt7dy36](https://tinyurl.com/gt7dy36)
  *MIT OCW*

- Grant Sanderson
  [https://goo.gl/R1kBdb](https://goo.gl/R1kBdb)
  *Essence of linear algebra, YouTube channel 3Blue1Brown*

- Think Different
  You can always find more to learn... If you want to ;-)