Problem Sheet

Answer ALL questions
Submission deadline: June 27, before 6 PM

1. You are given a basket of 80 identical balls, out of which all except one have the same weight. What is the minimum number of times you will have to use a given weighing balance before finding the odd ball in the basket?

2. Prove that for any arbitrary point P inside a triangle ABC,

$$AC + BC \geq AP + BP.$$

3. There are 100 passengers in a line waiting to board an air plane. Each of them hold a ticket to one of the 100 seats on the flight (for convenience, let us assume that the n^{th} passenger in the line has a ticket for seat number n). Unfortunately, the first person in the line is crazy and ends up occupying a randomly chosen seat. All other passengers however go to their assigned seat unless it is already occupied; if their seat is occupied, they occupy a randomly chosen free seat. What is the probability that the last (100^{th}) person to board the plane will sit in his assigned seat (seat number 100)?

4. Find all integers $x \neq y$, such that $x^y = y^x$.

5. You are given two identical eggs and have access to a 100-storey building. If an egg does not break on being dropped from a given floor of the building, it will survive a fall from all lower floors; similarly, if an egg breaks on being dropped from a floor, it will break when dropped from all higher floors. You are required to find the highest floor from which an egg on being dropped will not break. Note that an egg once broken cannot be reused; however, an egg which survives a fall can be used again. What will be your strategy to find a solution to this problem? Note that your strategy should require minimum number of egg drops; if there are n floors, what will be the number of egg drops required by your strategy in the worst case.

6. Consider n-length sequences of 0's and 1's. Out of all possible such sequences, determine the number of sequences that does not contain the pattern 1011 in them.

7. You are given two numbers (say, positive integers) with m and n digits respectively and need to transmit them through a communication channel from source to destination. The encoder at the source has to convert the given two numbers into one single number. On receiving the single number, the decoder at the destination needs to reconstruct the original two numbers exactly. What strategy will you use to encode the two numbers so that the length of the combined number that needs to be transmitted is as small as possible?

8. Prove that for any prime number $P > 5$ there exists K such that $\underbrace{111\ldots111}_{K \text{times}}$ (a decimal number made up of K consecutive ones) is divisible by P.

9. Assume you have n strings of length n. Give an efficient algorithm to sort the strings.

10. Some people in a village are infected with a deadly disease. The only symptom for the disease is a black spot on the infected person's forehead. A villager on knowing that he is infected will voluntarily leave the village so that the others do not get infected. Given that there are no mirrors or reflecting surfaces in the village, no infected person knows that he has the disease unless another person on seeing his forehead tells him so. If the people of the village decide to not inform any of the infected villagers of their disease, will the infected villagers ever get to know that they have the disease? If so, how many days will it take for all the infected villagers to leave the village.