# Algrabraic Complexity Theory

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# Overview

# Outline

## Computation without Bits

- An algorithm, in general, can use individual bits of the input in very complex ways. In particular, making execution decisions based on the values of a bit.
- Certain algorithms, however, use the individual bits in a much simpler way.
- Example: matrix multiplication. For $[c_{ij}] = [a_{ij}] \cdot [b_{ij}]$, we have:

$$c_{ij} = \sum_{k=0}^{n-1} a_{ik} b_{kj}.$$

- If we assume operations $+$ and $*$ as primitives, and the input being a sequence of numbers denoting entries of matrices, then the algorithm does not need to access bit values.

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  - Let $R$ be a ring with operations $+$ and $*$.
  - Let the input be variables $x_1$, $x_2$, ..., $x_n$.
  - An algorithm applies a sequence of ring operations on the input variables and constants from $R$.
  - The output is a polynomial in $R[x_1, x_2, \ldots, x_n]$.
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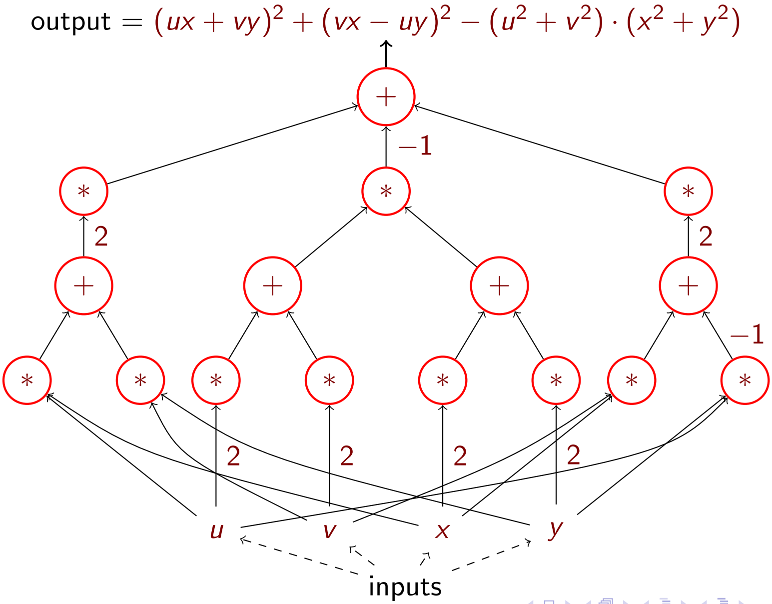
# An Example



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- As in the boolean settings, arithmetic circuit model is a non-uniform model of computation.
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# Power of the Model

- The model proposed by [Valiant 1979].
- It can compute all of the following operations:
    - Matrix operations: addition, multiplication, determinant, inverse, characteristic polynomial, permanent
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# Arithmetic Complexity

Crucial parameters associated with an arithmetic circuit are:

- Input length: number of input variables. Notice that the size of individual variables is not counted!
- Size: equals the number of operations in the circuit (measured as a function of input length).
- Depth: equals the length of the longest path from a variable to output of the circuit.
- Degree: equals the formal degree of circuit defined inductively as: 1 for input variables, max for addition gates, and sum for multiplication gates.
- Fanin: equals the largest number of inputs to a gate in the circuit. We allow arbitrary fanin.

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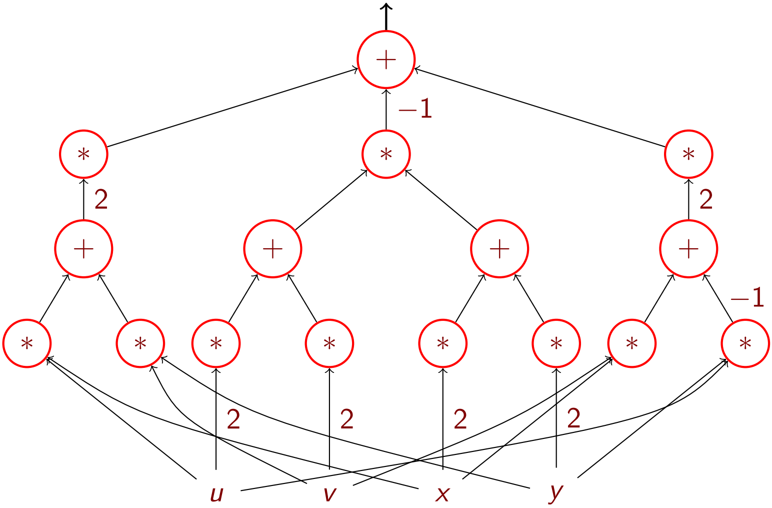
# Circuit Parameters




SIZE $= 16$ DEPTH $= 4$ DEGREE $= 4$ FANIN $= 3$

# Outline

1. Computation Over Rings
   - Arithmetic Circuit Model
   - Generalizing Arithmetic Circuits
2. Classes P and NP
3. Depth Reduction
4. Status of Lower Bounds
5. Polynomial Identity Testing
6. LPIT and Lower Bounds
7. Algorithms for 2-PIT and 3-PIT

# Extension with Zero-test

- Many other algebraic operations cannot be computed in arithmetic circuit model: solving system of linear equations, rank of a matrix, gcd of polynomials, primality testing ...
- Generalize the model by including another operation: zero-test.
  - This is a branching operation: check if the input is zero; if yes do $A$ else do $B$.
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# BSS Model

- The generalized model can still not compute simple functions, e.g., "Is $x < y$?"
- [Blum-Shub-Smale 1989] replaced zero-test with $\leq$ operator.
  - The operator makes sense only in rings with a total ordering, e.g., $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$.
- They showed that the model, for $R = \mathbb{Z}$ or $\mathbb{Q}$ restores access to bits, and is therefore equivalent to the standard boolean model.
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# Outline

# The Class P

- For both the models, the class P can be defined in an analogous way to boolean settings: all problems that can be solved by a circuit family of polynomial size.
- In the arithmetic circuit model, a problem is simply a family of polynomials, typically parameterized by the number of variables, or degree, or both:
  - Chebyshev polynomials

$$T_d(x) = \sum_{k=0}^{\lfloor d/2 \rfloor} \binom{d}{2k} (x^2 - 1)^k x^{d-2k}$$

    by degree,
  - Determinant polynomial by number of variables, and
  - Elementary symmetric polynomials

$$S_d(x_1, x_2, \ldots, x_n) = \sum_{I \subseteq [1,n], |I| = d} \prod_{j \in I} x_j,$$

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- In the arithmetic circuit model, the following problems are in P:
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# A Poor Definition of NP

- Analogous definition of NP to the boolean settings fails.
- Consider arithmetic circuit model, where each computation results in a polynomial, over $R = \mathbb{C}$.
- Say polynomial family $P_n(x_1, \ldots, x_n)$ is in NP if there exists another polynomial family $Q_{n+m+1}(x_1, \ldots, x_n, y_1, \ldots, y_m, z)$ in P such that:
  - $m = n^{O(1)}$, and
  - $P_n(\alpha_1, \ldots, \alpha_n) = \gamma$ iff there exists $\beta_1, \ldots, \beta_m$ with $Q_{n+m+1}(\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_m, \gamma) = 0$.

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- Therefore,

$$Q_{n+m+1}(\alpha_1, \ldots, \alpha_n, y_1, \ldots, y_m, z) = \delta \cdot (z - \gamma)^t,$$

  $t > 0$.
- Since this is true for all $\alpha_1$, ..., $\alpha_n$, we can reset $Q_{n+m+1}$ to $Q_{n+m+1}(\alpha_1, \ldots, \alpha_n, 0, \ldots, 0, z)$.

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# A Better Definition of NP

### The Class NP [Valiant 1979]

Polynomial family $\{P_n\}$ is in NP if there exists a family $\{P_{n+m}\} \in$ P such that $m = n^{O(1)}$, and for every $n$:

$$P_n(x_1, \ldots, x_n) = \sum_{y_1 \in \{0,1\}} \cdots \sum_{y_m \in \{0,1\}} Q_{n+m}(x_1, \ldots, x_n, y_1, \ldots, y_m).$$

1. Here 0 and 1 are identities of $R$.
2. The definition can be easily generalized to arithmetic circuit with zero-test model.

# Examples

- All problems in P,
- Permanent family,
- Jones polynomials: representing invariants of knots,
- Tutte polynomials:

$$T_G(x, y) = \sum_{A \subseteq E} (x-1)^{k(A)-k(E)} (y-1)^{k(A)+|A|-|V|}$$

where $G = (V, E)$ is an undirected graph and $k(A)$ is the number of connected components in the subgraph $(V, A)$.

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# NP-complete Problems

**Theorem [Valiant 1979]**

Computing permanent family is complete for NP in arithmetic circuit model: for every polynomial family $\{Q_n\}$ in NP, for every $n$, $Q_n$ can be expressed as permanent of a $n^{O(1)}$-size matrix with variable and constant entries.

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# Is P $\neq$ NP?

- The classes P and NP of arithmetic circuit model roughly correspond to computing the boolean classes #L and #P respectively:
  - Permanent is complete for #P in boolean model and for NP in arithmetic circuit model.
  - Determinant is complete for #L in boolean model and for P under quasi-polynomial size reductions in arithmetic circuit model.
- Therefore, it is a weaker question that P $\neq$ NP in boolean model: If P $\neq$ NP in boolean model then P $\neq$ NP in arithmetic circuit model.

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- Even for arithmetic circuit model, proving P $\neq$ NP has been very challenging, and has remained a hypothesis.
- Henceforth, we restrict ourselves to the arithmetic model of computation.
- For arithmetic circuit model, the classes P and NP are called VP and VNP: named after Valiant.
- Over the years, this problem has become one of the most active areas of research in complexity theory.

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# Outline

# Reducing Depth to $O(\log d)$

**Theorem (Valiant-Skyum-Berkowitz-Rackoff, 1983)**

*If polynomial $P(x_1, \ldots, x_n)$ of degree $d$ is computable by an arithmetic circuit of size $s \geq n$, then it can also be computed by an arithmetic circuit of size $s^{O(1)}$ whose depth is $O(\log d)$ and fanin of multiplication gates is two.*

Another construction was given by [Allender-Jiao-Mahajan-Vinay 1994].

# Reducing Depth to 4

**Theorem (A-Vinay 2008)**

*If polynomial $P(x_1, \ldots, x_n)$ of degree $d$ is computable by an arithmetic circuit of size $s = 2^{o(d + d\log\frac{n}{d})}$, then it can also be computed by an arithmetic circuit of size $s^{O(1)}$ of depth $4$.*

Extended by [Koiran 2012, Tavenas 2013].

# Proof

Let the polynomial $P(x_1, \ldots, x_n)$ be computed by an arithmetic circuit $C$ of size $t = 2^{o(d + d\log\frac{n}{d})}$.

- [Allender-Jiao-Mahajan-Vinay 1994] shows that $C$ can be transformed to a circuit $D$ of degree $d$, size $t^{O(1)}$ and depth $O(\log d)$ with multiplication gates of fanin two.
- We modify this transformation slightly to obtain a circuit $D$ of degree $d$, size $t^{O(1)}$ and depth $\leq 2\log d$ with multiplication gates of fanin $\leq 6$.
- Further, the circuit $D$ consists of alternating layers of addition and multiplication gates.
- We now describe the construction of the circuit $D$.

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# Construction of $D$: Setup

- Make the circuit $C$ layered with alternating layers of addition and multiplication gates.
- Make fanin of every multiplication gate two.
- Rearrange children of multiplication gates so that degree of the right child is greater than or equal to the degree of the left child.

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# Construction of $D$: Proof Trees

A proof tree rooted at gate $g$ of circuit $C$ is a subcircuit of $C$ obtained as follows:

- Start with the subcircuit of $C$ that has gate $g$ at the top and computes the polynomial at gate $g$.
- For every $+$-gate in the subcircuit, retain only one input to the gate deleting the remaining input lines.
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A proof tree rooted at gate $g$ computes a monomial and the polynomial at $g$ is the sum over monomials computed by all proof tress rooted at $g$.

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# Construction of $D$: Defining Intermediate Polynomials

- For every input variable $x_i$, let $[x_i]$ stand for the polynomial $x_i$.
- For every gate $g$ of $C$, let $[g]$ stand for polynomial computed at gate $g$.
- For every pair of gates $g$ and $h$ of $C$, let $[g, h]$ be the polynomial:

$$[g, h] = \sum_T m(T, h)$$

  where $T$ runs over all proof trees rooted at $g$ and $m(T, h)$ is the monomial computed by proof tree $T$ when gate $h$ is replaced by $1$ if gate $h$ occurs in the rightmopst path of $T$, $m(T, h)$ is $0$ otherwise.
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- Let $g$ be a $*$-gate with children $g_L$ (left child) and $g_R$ (right child).
- A rightmost path from $g$ to $h$ is a path from $g$ to $h$ in the circuit obtained from $C$ by deleting input line from left child of every $*$-gate.
- If there are only $+$-gates on every rightmost path from $g$ to $h$ then

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- Otherwise, there exists a $*$-gate $p$ with children $p_L$ and $p_R$ in a rightmost path from $g$ to $h$ such that $\deg(p) \geq \frac{1}{2}(\deg(g) + \deg(h)) > \deg(p_R)$.
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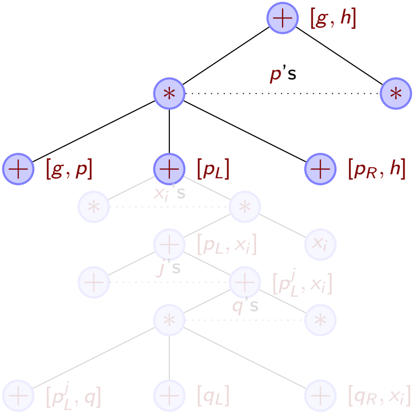


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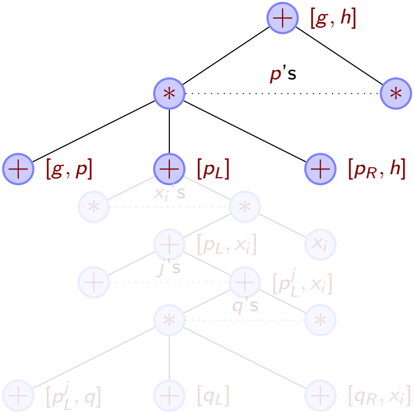


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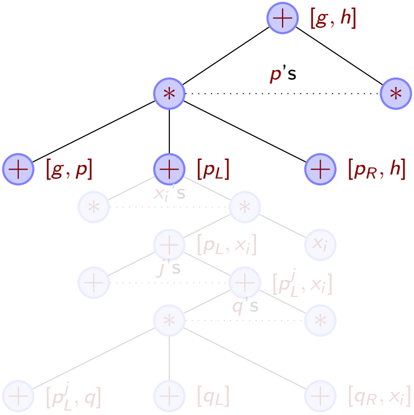


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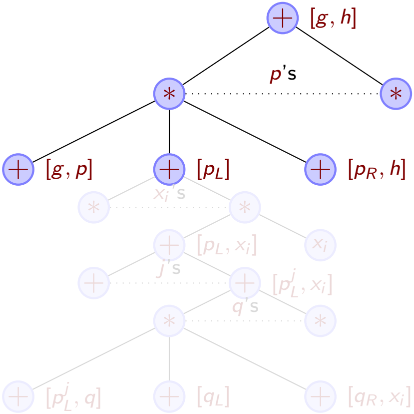


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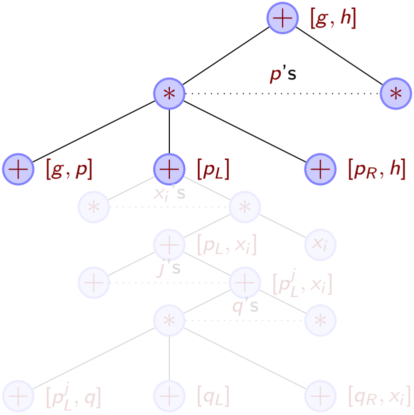


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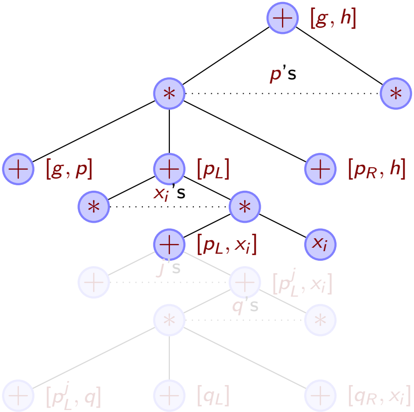


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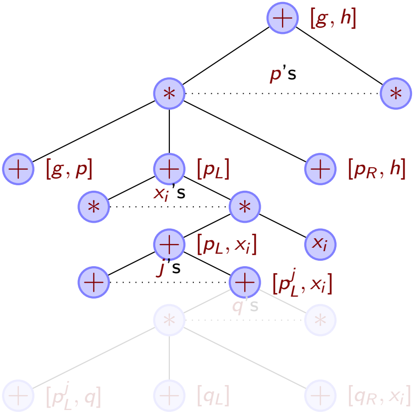


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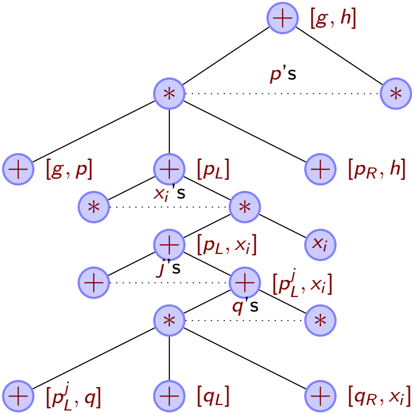


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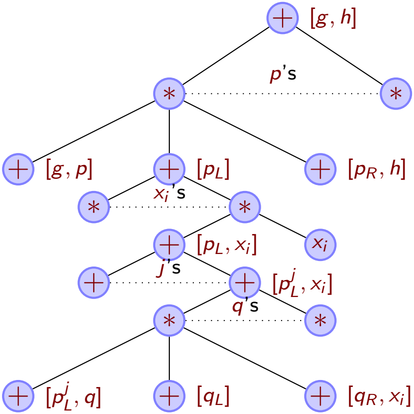


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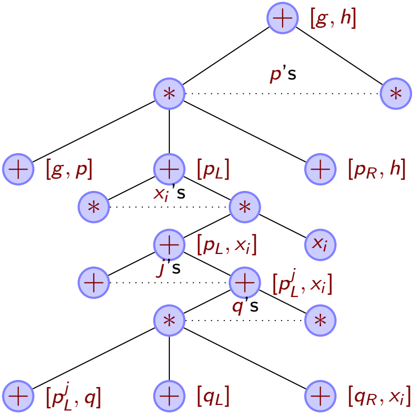


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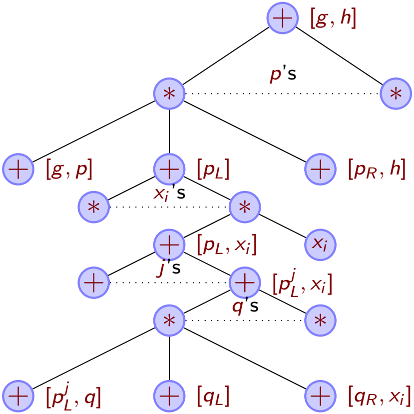


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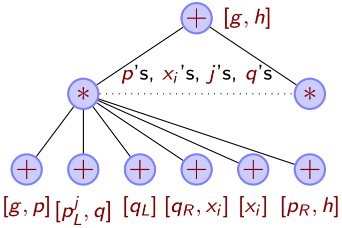


Flatten the subcircuit to write $[g,h]$ as:

$$[g,h] = \sum_p \sum_i \sum_j \sum_q [g,p][p_{L,j},q][q_L][q_R,x_i][x_i][p_R,h]$$

with degree of each of the six polynomials in the product bounded by $\frac{1}{2}\deg([g,h])$.

# Construction of $D$

- By adding dummy $+$-gates and merging adjacent $+$-gates, it can be ensured that the circuit has alternating layers of $+$- and $*$-gates.
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# Cutting $D$

- Let $\ell$ be any function such that $\ell \leq \frac{d + d \log \frac{d}{n}}{\log t}$ and $\ell = \omega(1)$.
- Let $u = \frac{1}{2} \log_6 \ell$.
- Cut $D$ into two halves with top half consisting of $u$ layers of $*$-gates with the bottom layer being of $*$-gates.
- Let $g_1$, $g_2$, ..., $g_k$ be the output gates of the bottom layer.
- Let the polynomial computed by gate $g_i$ be $P_i(x_1, x_2, \ldots, x_n)$.
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$$P(x_1, \ldots, x_n) = P_0(P_1(x_1, \ldots, x_n), P_2(x_1, \ldots, x_n), \ldots, P_k(x_1, \ldots, x_n)).$$

# Cutting $D$

- Let $\ell$ be any function such that $\ell \leq \frac{d + d \log \frac{d}{n}}{\log t}$ and $\ell = \omega(1)$.
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# The Circuit $E$

- A direct counting shows that each $P_j$, $0 \leq j \leq k$, can be replaced by a depth two circuit of size $2^{o(d + d \log \frac{n}{d})}$.
- Since $k = 2^{o(d + d \log \frac{n}{d})}$, the resulting depth four circuit, $E$, is of size $2^{o(d + d \log \frac{n}{d})}$.
- The fanin of second layer of $*$-gates in $E$ is at most $6^u = \sqrt{\ell}$ which is any small function in $\omega(1)$.
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# Reducing Depth to 3

**Theorem (Gupta-Kamath-Kayal-Saptharishi 2013)**

*If polynomial $P(x_1, \ldots, x_n)$ of degree $d$ is computable by an arithmetic circuit of size $s = 2^{o(d + d \log \frac{n}{d})}$, then it can also be computed by an arithmetic circuit of size $s^{O(1)}$ of depth $3$ if the underlying field has characteristic zero or large $(= \Omega(\log s))$.*

# Proof Outline

- Replace each $\prod$ layer of a depth four circuit by $\sum \wedge \sum$ layers resulting in a $\sum \wedge \sum \wedge \sum$ circuit using [Fischer 1994]:

$$\prod_{j=1}^{n} x_j = \frac{1}{2^{n-1} n!} \sum_{r_2,\ldots,r_n \in \{-1,1\}} (-1)^{wt(r)} (x_1 + \sum_{j=2}^{n} r_j x_j)^n,$$

  where $wt(r) = |\{j \mid r_j = -1\}|$. This works for char $= 0$ or $> n$.

- Replace $\wedge \sum \wedge$ by $\sum \prod \sum$ resulting in $\sum \prod \sum$ circuit using [Saxena 2008]:

$$(\alpha_1 x_1^{\beta_1} + \alpha_2 x_2^{\beta_2} + \cdots + \alpha_n x_n^{\beta_n})^d = \text{degree } d \text{ coefficient of } d! \cdot \prod_{j=1}^{n} e^{\alpha_j x_j^{\beta_j} z}.$$

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# Outline

[Jerrum-Snir 1982] Any monotone circuit family computing permanent is of exponential size.

- Monotone circuits are circuits with no negative constant.

[Shpilka-Wigderson 1999] Any depth three circuit family computing permanent (or even determinant) over $\mathbb{Q}$ is of size $\Omega(n^2)$.

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[Raz 2004] Any multilinear formula family computing permanent or determinant is of size $n^{\Omega(\log n)}$.

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# Outline

# Definitions

### PIT

Given an arithmetic circuit of size $s$ over ring $R$, test if the polynomial computed by the circuit is non-zero.

### Low Degree PIT (LPIT)

Given an arithmetic circuit of size $s$ over ring $R$ computing a polynomial of degree $\leq s$, test if the polynomial computed by the circuit is non-zero.

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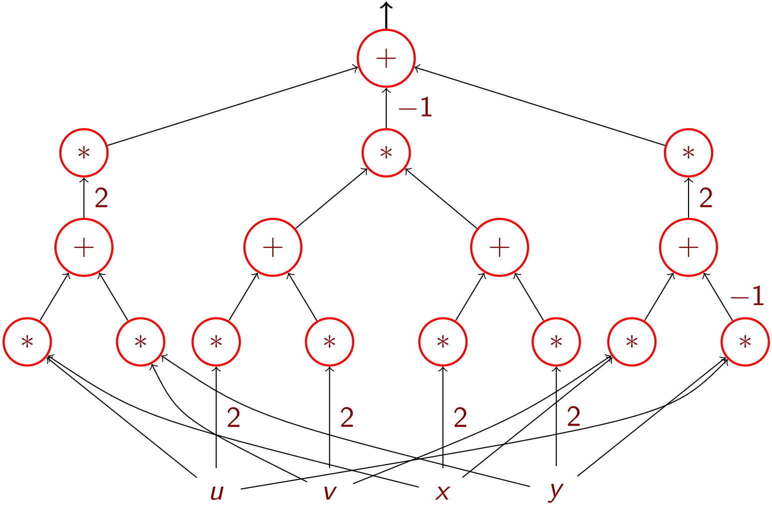
# An Example

Is $(ux + vy)^2 + (vx - uy)^2 - (u^2 + v^2) \cdot (x^2 + y^2) \neq 0$? [NO!]



# Applications

Bipartite Matching : for graph $G = (U, V, E)$, check if

$$\det \begin{bmatrix} e_{1,1}x_{1,1} & \cdots & e_{1,n}x_{1,n} \\ \vdots & \ddots & \vdots \\ e_{n,1}x_{n,1} & \cdots & e_{n,n}x_{n,n} \end{bmatrix} \neq 0$$

over any field, where $E = [e_{i,j}]$. An example of LPIT.

Primality Testing : for number $n$, check if

$$(x + y)^n = x^n + y^n$$

over ring $Z_n[x, y]/(x^r - 1, y^s - 1)$ for suitable $r$ and $s$, both $\log^{O(1)} n$.

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# Complexity of PIT

A number of randomized polynomial time algorithms are known for the problem.

- The simplest one is by [Schwartz, Zippel 1979]: Substitute random values from a small subset of $R$ (using a small extension of $R$ if required) for each variable, evaluate the circuit, and output NON-ZERO iff the result is a non-zero number.
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# Deterministic Algorithm for PIT

## Open Question

Is there a deterministic polynomial time algorithm for PIT?

- Long-standing open problem.
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# Two Types of Deterministic Algorithms for PIT

### White Box

A white-box time $t(n)$ algorithm for PIT is a deterministic algorithm solving the problem in time at most $t(n)$.

### Black Box

A black-box time $t(n)$ algorithm for PIT is a deterministic algorithm running in time $t(n)$ that, given an arithmetic circuit, determines if it computes non-zero polynomial with access only to input-output lines and size of the circuit.

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# Outline

# LPIT and Lower Bounds I

**Theorem (Kabanets-Impagliazzo 2003)**

*If there exists a white-box polynomial-time algorithm for LPIT then NEXP requires superpolynomial size arithmetic circuits.*

# LPIT and Lower Bounds I

**Proof.**

- Assume NEXP has polynomial-size arithmetic circuits and PIT has a polynomial-time algorithm.
- Construct an NP machine to compute permanent that guesses the circuit for the permanent and verifies it recursively using PIT:
  - If $C(x_{1,1},\ldots,x_{1,n},\ldots,x_{n,1},\ldots,x_{n,n})$ is circuit for permanent of $n \times n$ matrices, then we can extract from it circuit $C_j$ for permanent of $j \times j$ matrices for $j < n$.
  - Using LPIT, verify the correctness of $C$:

$$C_j(\bar{x}) = x_{1,1}C_{j-1}(\bar{x}_1) + \cdots + x_{1,j}C_{j-1}(\bar{x}_j)$$

    where $\bar{x}_i$ drops first row and $i$th column.
- This implies #P is in NP. Since NEXP = #P by assumption, we get NEXP = NP contradicting time hierarchy theorem.

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# LPIT and Lower Bounds II

**Theorem (Heintz-Schnorr 1980, A 2005)**

*If there exist a black-box polynomial-time algorithm for LPIT then E requires exponential size arithmetic circuits.*

# LPIT and Lower Bounds II

## Proof.

- Let $\mathcal{A}$ be a black-box polynomial-time algorithm for LPIT.
- For a circuit of size $s$ on $n$ variables, $\mathcal{A}$ will evaluate it on a sequence of inputs and accept iff any of the outputs in non-zero.
- Let these inputs be $(\alpha_{1,1}, \ldots, \alpha_{1,n}), \ldots, (\alpha_{t,1}, \ldots, \alpha_{t,n})$ with $t = s^{O(1)}$.
- Let $m = \lceil \log(t+1) \rceil = O(\log s)$.
- Define polynomial $r_m$ as:

$$r_m(x_1, x_2, \ldots, x_m) = \sum_{S \subseteq [1,m]} c_S \prod_{i \in S} x_i.$$

- Coefficients $c_S \in F$ satisfy:

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for every $1 \leq j \leq t$.

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- A non-zero $r_m$ always exists since it has $\geq t+1$ coefficients that satisfy $t$ homogeneous linear equations.
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# Fixed Depth PIT

### Depth $d$ PIT

$d$-PIT is the problem to decide if a given arithmetic circuit of depth $d$ (alternating sums and products with top gate being sum) computes a non-zero polynomial.

$d$-PIT is a restriction of LPIT.

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# 3-PIT and Lower Bounds

**Theorem (Gupta-Kamath-Kayal-Saptharishi 2013)**

*If there exist a polynomial-time black-box algorithm for 3-PIT then $E$ requires exponential size arithmetic circuits if the underlying field has characteristic zero or large ($= \Omega(\log s)$).*

**Theorem**

*If there exists a white-box polynomial-time algorithm for 3-PIT then $NEXP$ requires superpolynomial size arithmetic circuits.*

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# Outline

# 2-PIT

**Theorem (Folklore)**

*There exists a polynomial-time black-box algorithm for 2-PIT.*

# 2-PIT

**Proof.**

- A $\sum\prod$ circuit computes a sparse polynomial.
- Let $C$ be the given $\sum\prod$ circuit of size $s$ computing a polynomial of degree $\leq d$.
- One of the substitutions $(x_1, \ldots, x_i, \ldots, x_n) = (y, \ldots, y^{(d+1)^{i-1} \pmod r}, \ldots, y^{(d+1)^{n-1} \pmod r})$, $1 < r < s^2$, will ensure that all terms of the polynomial remain distinct.

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# 3-PIT with Bounded Top Fanin

Sequence of solutions for 3-PIT with top sum gate of fanin $k$:

[Dvir-Shpilka 2005] White-box $2^{(\log s)^{k^2}}$ time algorithm.

[Kayal-Saxena 2006] White-box $s^{O(k)}$ time algorithm.

[Karnin-Shpilka 2008] Black-box $s^{O(\log^k s)}$ time algorithm.

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# Jacobian Based Algorithm

- Let $P = \sum_{i=1}^{k} T_i$, $T_i = \prod_{j=1}^{s} L_{i,j}$ be the given circuit with $L_{i,j} = \alpha_{i,j,0} + \sum_{\ell=1}^{n} \alpha_{i,j,\ell} x_\ell$.
- Assume that $P \neq 0$ and $T_i$'s are algebraically independent:
  - There is no polynomial $Q(y_1, y_2, \ldots, y_k)$ such that $Q(T_1, T_2, \ldots, T_k) = 0$.
- For characteristic zero or $> s^k$: $T_1, \ldots, T_k$ are algebraically independent iff $J(T_1, T_2, \ldots, T_k)$ has full rank, where

$$J(y_1, y_2, \ldots, y_k) = \begin{bmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \cdots & \frac{\partial y_1}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial y_k}{\partial x_1} & \frac{\partial y_k}{\partial x_1} & \cdots & \frac{\partial y_k}{\partial x_n} \end{bmatrix}.$$

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- Therefore, $J(T_1, \ldots, T_k)$ has rank $k$.
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&= \begin{bmatrix} T_1 \sum_{j=1}^{d} \frac{\alpha_{1,j,1}}{L_{1,j}} & \cdots & T_1 \sum_{j=1}^{d} \frac{\alpha_{1,j,n}}{L_{1,j}} \\ \vdots & \ddots & \vdots \\ T_k \sum_{j=1}^{d} \frac{\alpha_{k,j,1}}{L_{k,j}} & \cdots & T_k \sum_{j=1}^{d} \frac{\alpha_{k,j,n}}{L_{k,j}} \end{bmatrix}
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# Jacobian Based Algorithm

- Assume, wlog, that columns corresponding to variables $x_1, x_2, \ldots, x_k$ have rank $k$.
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$$\begin{aligned}
\hat{P} &= \begin{vmatrix} T_1 \sum_{j=1}^{d} \frac{\alpha_{1,j,1}}{L_{1,j}} & \cdots & T_1 \sum_{j=1}^{d} \frac{\alpha_{1,j,k}}{L_{1,j}} \\ \vdots & \ddots & \vdots \\ T_k \sum_{j=1}^{d} \frac{\alpha_{k,j,1}}{L_{k,j}} & \cdots & T_k \sum_{j=1}^{d} \frac{\alpha_{k,j,k}}{L_{k,j}} \end{vmatrix} \\
&= \prod_{i=1}^{k} T_i \cdot \begin{vmatrix} \sum_{j=1}^{d} \frac{\alpha_{1,j,1}}{L_{1,j}} & \cdots & \sum_{j=1}^{d} \frac{\alpha_{1,j,k}}{L_{1,j}} \\ \vdots & \ddots & \vdots \\ \sum_{j=1}^{d} \frac{\alpha_{k,j,1}}{L_{k,j}} & \cdots & \sum_{j=1}^{d} \frac{\alpha_{k,j,k}}{L_{k,j}} \end{vmatrix} \\
&= \prod_{i=1}^{k} T_i \cdot R,
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where $R$ is a sparse rational function.

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- Since $\hat{P}$ is a product of sparse polynomials and rational functions, the set of substitutions as used for 2-PIT will ensure that $\hat{P}$ remains non-zero under one of them.
- For this substitution, the Jacobian has full rank and therefore the circuit output remains non-zero.

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# 3-PIT for Diagonal Circuits

**Diagonal Circuits**

Circuits where each multiplication gate is a powering gate.

**Theorem (Forbes-Shpilka 2012, A-Saha-Saxena 2013)**

*There exists a $s^{O(\log s)}$-time black-box algorithm for diagonal 3-PIT.*

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# Rank Concentration Based Algorithm

- Let $P = \sum_{i=1}^{k} T_i$, $T_i = L_i^d$ be the given circuit with $L_i = \alpha_{i,0} + \sum_{\ell=1}^{n} \alpha_{i,\ell} x_\ell$.
- The polynomial can be rewritten as:

$$P = \bar{1} \cdot (\bar{u}_0 + \bar{u}_1 x_1 + \cdots + \bar{u}_n x_n)^d,$$

where $\bar{u}_\ell = [\alpha_{1,\ell} \cdots \alpha_{k,\ell}]$.

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- Now consider the following polynomial with vectors over $F^k$ as coefficients:

$$\begin{aligned} Q &= (\bar{u}_0 + \bar{u}_1 x_1 + \cdots + \bar{u}_n x_n)^d \\ &= \sum_{S \in [0,d]^n} \bar{v}_S \bar{x}^S \end{aligned}$$

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- $\ell$-rank concentration is the property that $\bar{v}_S$ of support $\ell$ (i.e., $S$ with only $\ell$ non-zero $d_i$'s) span this space.
- If there is $\ell$-rank concentration, the PIT can be solved by setting all but $\ell$ $x$'s to zero and evaluating the resulting polynomial.

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- The space spanned by $\bar{v}_S$ has $\log m$-rank concentration:
  - Consider a monomial $\bar{x}^S$ with support $> \log m$. It has $> m$ monomials strictly below it in lex-ordering.
  - There must be linear dependence between coefficients associated with these lower monomials.
  - Define a total ordering on monomials by fixing an arbitrary order between variables.
  - Take a linear dependence equation for lower monomial coefficients, identify the largest monomial in total order, and multiply the equation with coefficient of a monomial such that the largest monomial becomes $\bar{x}^S$.
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- The algorithm is now simple: for all subsets of $\log m$ variables, set the remaining variables to zero, and test if the resulting polynomial is zero on $d^{\log m}$ distinct values.
- This gives a $d^{O(\log d)}$-time black-box algorithm.
- In certain situations, there may not be rank concentration to begin with.
- So first apply a transformation on variables that yields rank concentration.
- For certain other restrictions of 3-PIT, the following transformation works:

$$x_i \mapsto x_i + t^{d_i}$$

for small $d_i$'s.

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