Introduction to Polyhedral Model

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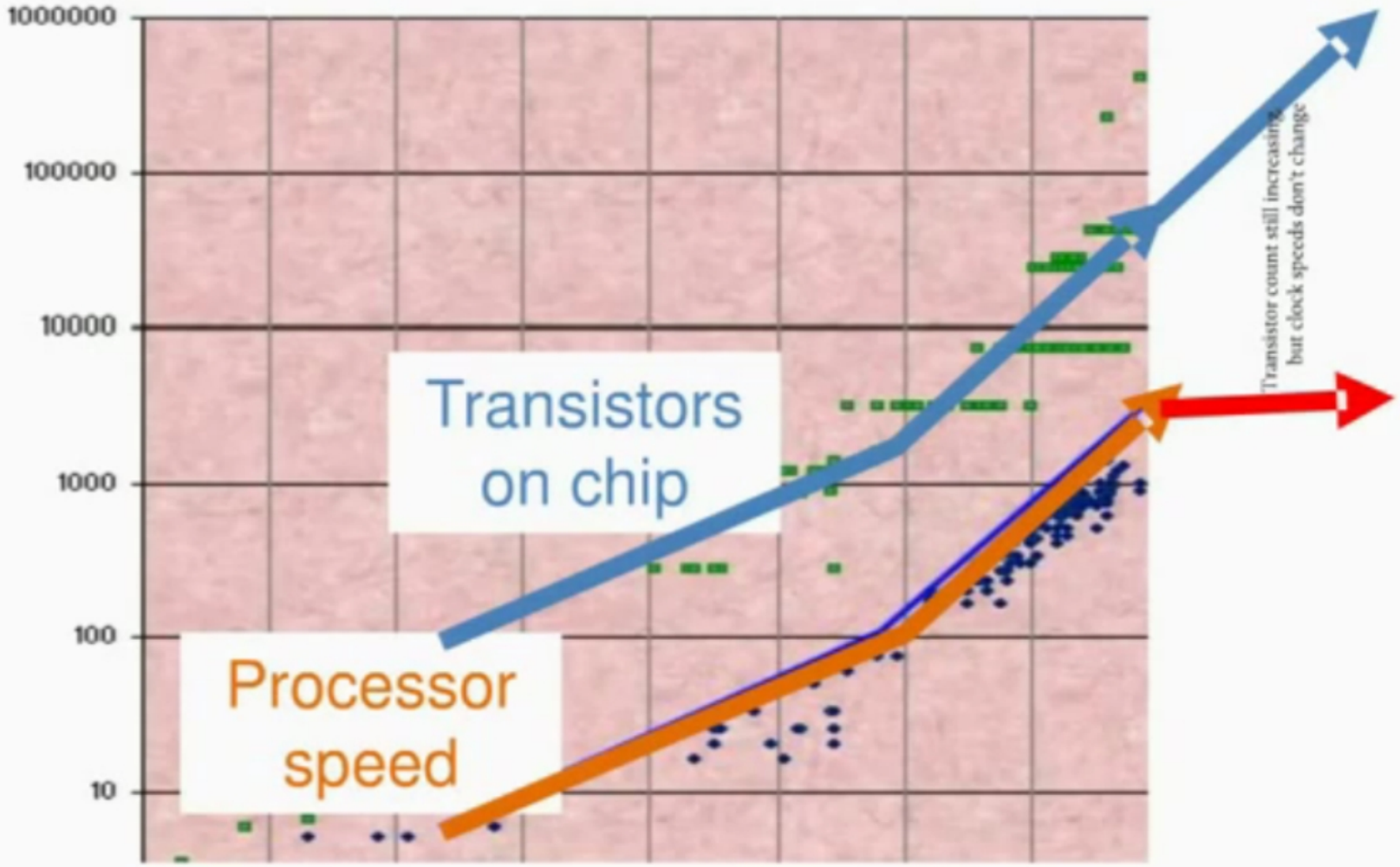
Moore's Law


1000000
100000
10000
1000
100
10
Transistors on chip
Processor speed
Transistor count still increasing
but clock speeds don't change

Speed up with multicores

Example

```
for (i = 0; i < N; ++i)
  for (j = 0; j < N; ++j)
    A[i][j] = i * j;
```

Speed up with multicores

Example

```
for (i = 0; i < N; ++i)
  for (j = 0; j < N; ++j)
    A[i][j] = i * j;
```

Example

```
#pragma omp parallel
for (i = 0; i < N; ++i)
  for (j = 0; j < N; ++j)
    A[i][j] = i * j;
```

Speed up with multicores

Example

```
for (i = 0; i < N; ++i)
  for (j = 0; j < N; ++j)
    A[i][j] = A[i][j-1]+ 3;
```

Example

```
for (i = 0; i < N; ++i)
  for(j = 0; j < N; ++j)
   A[i][j]=(A[i-1][j-1]+A[i-1][j]+A[i][j-1]) / 3;
```

How to Exploit Parallelism?

- **Parallel Programming is hard**
- Can a Compiler automatically parallelize the serial code?
- Parallelization of Regular scientific applications
- Most of time is spent in regular for loops
- Arrays as main data structure
- **Polyhedral Model** is a mathematical, high level IR of program, which helps in reasoning about parallelism

Motivating Example [1/2]

Example

```
for (i = 0; i < 2; ++i)
  for (j = 0; j < 3; ++j)
    A[i][j] = i * j;
```

Program execution:

```
1: A[0][0] = 0 * 0;
2: A[0][1] = 0 * 1;
3: A[0][2] = 0 * 2;
4: A[1][0] = 1 * 0;
5: A[1][1] = 1 * 1;
6: A[1][2] = 1 * 2;
```

Motivating Example [2/2]

A few observations:

- Statement is executed 6 times
- There are different values of i, j associated with these 6 instances
- There is an order on them (the execution order)

Objective:
Construct a representation where these 3 characteristics are captured

Polyhedral Representation

From syntactic code to polyhedral representation

- Modeling iteration domains with polytopes

Underlying mathematical concepts:

- Affine functions, hyperplanes, affine transformations
- Polyhedra (bounded, rational, integer and parametric)

Definition (**Linear**)

A k-dimensional function f is linear iff it can expressed in the following form:

$$linear \quad function \quad f(\vec{v}) = M_f \vec{v} \tag{1}$$

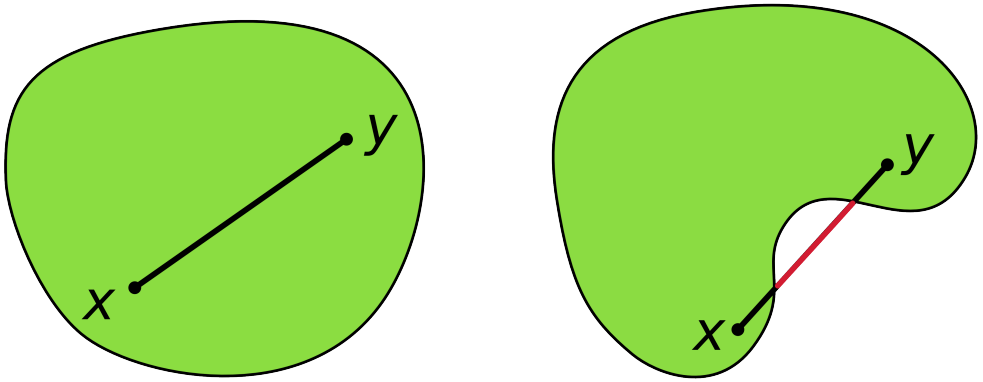
Definition (**Affine**)

A k-dimensional function f is affine iff it can expressed in the following form:

$$affine \quad function \quad f(\vec{v}) = M_f \vec{v} + \vec{f_0} \tag{2}$$

Definition (**Affine sub-space**)

A set of vectors is an affine sub-space of a vector space iff it is closed under affine combination, i.e., if $\vec{x}, \vec{y}$ are in the space, all points lying on the line joining $\vec{x}$ and $\vec{y}$ belong to the space.



Definition (**Affine hyperplane**)

An affine hyperplane is an $n-1$ dimensional affine sub-space of an n dimensional space.

$$\{\vec{v} \quad | \quad \mathbf{h}.\vec{v} = k\}$$

Polyhedron

Definition (Polyhedron)

A set $\mathcal{S} \in \mathbb{K}^m$ is a polyhedron if there exists a system of a finite number of affine inequalities $A\vec{x} \leq \vec{b}$ such that:

$$\mathcal{P} = \{\vec{x} \in \mathbb{K}^m \mid A\vec{x} \leq \vec{b}\}$$

Equivalently, it is the intersection of finitely many half-spaces.

Definition (Polytope)

A polytope is a bounded polyhedron.

Overview of the Polyhedral Representation

- Iteration domain: set of totally ordered n-dimensional vectors
  - **Iteration vector** $\vec{x}_S = (i, j)$
  - Iteration domain: the set of values of $\vec{x}_S$
- Convenient approach: **polytopes model sets** of totally ordered n-dimensional vectors

Returning to the Example

Modeling the iteration domain:

- Polytope dimension: set by the number of surrounding loops
- Constraints: set by the loop bounds

$$\mathcal{D}_R : \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix} \cdot \begin{pmatrix} i \\ j \end{pmatrix} + \begin{pmatrix} 0 \\ 2 \\ 0 \\ 2 \end{pmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & -1 & 2 \end{bmatrix} \cdot \begin{pmatrix} i \\ j \\ 1 \end{pmatrix} \geq \vec{0}$$

$$0 \leq i \leq 2, \quad 0 \leq j \leq 2$$

Increasing the Expressiveness

Problems:

- Parametric loop bounds: use parametric polyhedra!

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \end{bmatrix} \cdot \begin{pmatrix} i \\ j \\ N \\ 1 \end{pmatrix} \geq \vec{0}$$

$$0 \leq i \leq N, \quad 0 \leq j \leq N$$

Increasing the Expressiveness

Problems:

- Non-unit loop strides: normalize the loop!

Example

```
for (i = 0; i < N; i = i+5)
  A[i] = i + 10;
```

Example

```
for (i = 0; i < N/5; i++)
  A[i*5] = (i*5) + 10;
```

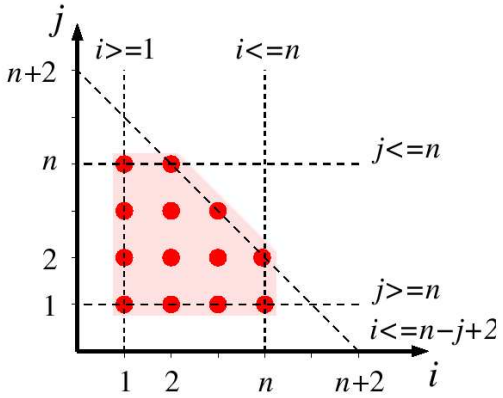
Increasing the Expressiveness

Problems:

- Conditionals:
  - Those which preserve convexity: ok! (add affine constraints)
  - Problem remains for the others...
  - Same can done for while loops which affine exit condition

```
for (i=1; i<=n; ++i)
. for (j=1; j<=n; ++j)
. . if (i<=n-j+2)
. . . s[i] = ...
```

$$\mathcal{D}_{S1} = \begin{bmatrix} 1 & 0 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ -1 & -1 & 1 & 2 \end{bmatrix} \cdot \begin{pmatrix} i \\ j \\ n \\ 1 \end{pmatrix}$$



Iteration domain of S_1

Affine schedule

Definition (Affine schedule)

Given a statement S, a p-dimensional affine schedule $\mathbf{\Theta^S}$ is an affine form on the outer loop iterator $\vec{x}_S$ and the global parameters $\vec{n}$. It is written:

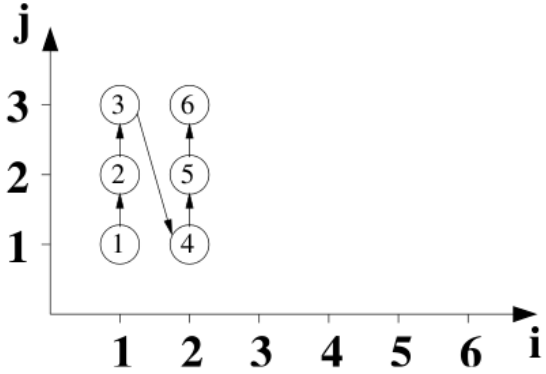
$$\mathbf{\Theta^S}(\vec{x}_S) = T_S \begin{pmatrix} \vec{x}_S \\ \vec{n} \\ 1 \end{pmatrix}, T_S \in \mathbb{K}^{p \times dim(\vec{x}_S) + dim(\vec{n}) + 1} \tag{3}$$

- **A schedule assigns a timestamp to each executed instance of a statement**
- Lexicographic ordering of timestamps

Scheduling Statement Instances Example[1/3]

Interchange Transformation

The transformation matrix is the identity with a permutation of two rows.



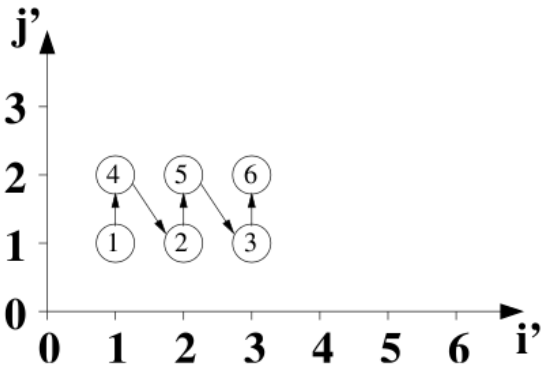
$$\begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix} \begin{pmatrix} i \\ j \end{pmatrix} + \begin{pmatrix} -1 \\ 2 \\ -1 \\ 3 \end{pmatrix} \geq \vec{0}$$

(a) original polyhedron
$A\vec{x} + \vec{a} \geq \vec{0}$

$\Longrightarrow$

$$\begin{pmatrix} i' \\ j' \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{pmatrix} i \\ j \end{pmatrix}$$

(b) transformation function
$\vec{y} = T\vec{x}$



$$\begin{bmatrix} 0 & 1 \\ 0 & -1 \\ 1 & 0 \\ -1 & 0 \end{bmatrix} \begin{pmatrix} i' \\ j' \end{pmatrix} + \begin{pmatrix} -1 \\ 2 \\ -1 \\ 3 \end{pmatrix} \geq \vec{0}$$

(c) target polyhedron
$(AT^{-1})\vec{y} + \vec{a} \geq \vec{0}$

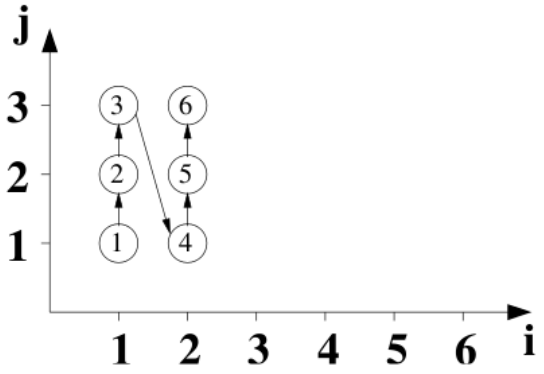
```
do i = 1, 2
  do j = 1, 3
```

```
do j = 1, 3
  do i = 1, 2
```

Scheduling Statement Instances Example[2/3]

Reversal Transformation

The transformation matrix is the identity with one diagonal element replaced by -1.



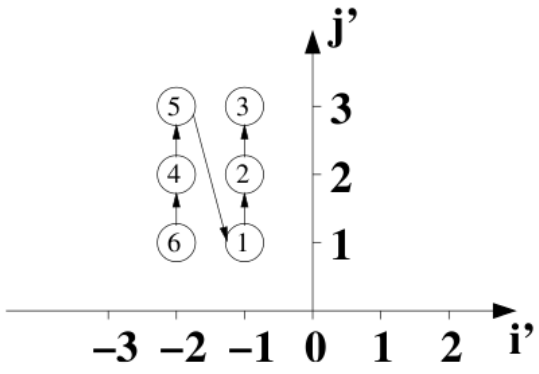
$$\begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix} \begin{pmatrix} i \\ j \end{pmatrix} + \begin{pmatrix} -1 \\ 2 \\ -1 \\ 3 \end{pmatrix} \geq \vec{0}$$

(a) original polyhedron
$A\vec{x} + \vec{a} \geq \vec{0}$

$$\Longrightarrow$$

$$\begin{pmatrix} i' \\ j' \end{pmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{pmatrix} i \\ j \end{pmatrix}$$

(b) transformation function
$\vec{y} = T\vec{x}$



$$\begin{bmatrix} -1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix} \begin{pmatrix} i' \\ j' \end{pmatrix} + \begin{pmatrix} -1 \\ 2 \\ -1 \\ 3 \end{pmatrix} \geq \vec{0}$$

(c) target polyhedron
$(AT^{-1})\vec{y} + \vec{a} \geq \vec{0}$

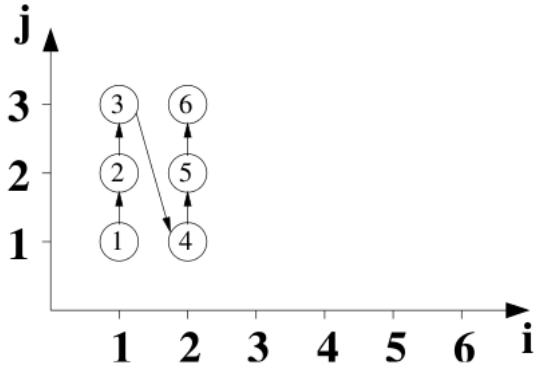
```
do i = 1, 2
  do j = 1, 3
```

```
do i = -1, -2, -1
  do j = 1, 3
```

Scheduling Statement Instances Example[3/3]

Coumpound Transformation

The transformation matrix is the composition of an interchange and reversal



$$\begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix} \begin{pmatrix} i \\ j \end{pmatrix} + \begin{pmatrix} -1 \\ 2 \\ -1 \\ 3 \end{pmatrix} \geq \vec{0}$$

(a) original polyhedron

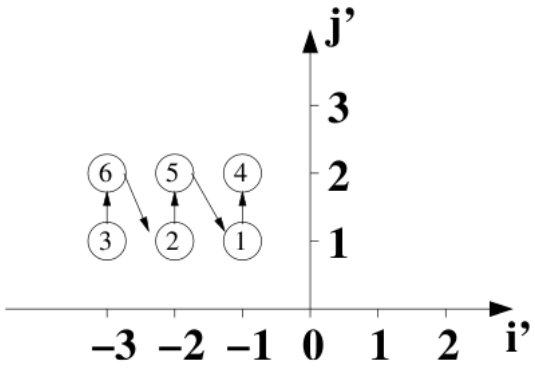
$A\vec{x} + \vec{a} \geq \vec{0}$

$\Longrightarrow$

$$\begin{pmatrix} i' \\ j' \end{pmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{pmatrix} i \\ j \end{pmatrix}$$

(b) transformation function

$\vec{y} = T\vec{x}$



$$\begin{bmatrix} 0 & -1 \\ 0 & 1 \\ 1 & 0 \\ -1 & 0 \end{bmatrix} \begin{pmatrix} i' \\ j' \end{pmatrix} + \begin{pmatrix} -1 \\ 2 \\ -1 \\ 3 \end{pmatrix} \geq \vec{0}$$

(c) target polyhedron

$(AT^{-1})\vec{y} + \vec{a} \geq \vec{0}$

```
do i = 1, 2
  do j = 1, 3
```

```
do j = -1, -3, -1
  do i = 1, 2
```

Are all Transformations Valid?

Example

```
for (i = 1; i < N; ++i)
  A[i] = (A[i-1] + A[i] ) / 2;
```

Is Loop Reversal valid?

Data Dependence

Definition (Conditions)

Given two iterations, there exists a dependence between them if the following conditions are met:

- they reference the same memory location
- one of the accesses is a write

Three categories of dependences:

- RAW (Read-After-Write, aka flow): first reference writes, second reads
- WAR (Write-After-Read, aka anti): first reference reads, second writes
- WAW (Write-After-Write, aka output): both references writes

Another kind: RAR (Read-After-Read dependences), used for locality/reuse computations

Illustrations

Example

```
for (i = 0; i < N; ++i)
  for (j = 0; j < N; ++j)
    A[i][j] = A[i-1][j-1];
```

Example

```
for (i = 0; i < N; ++i)
  for(j = 0; j < N; ++j)
   A[i][j]=(A[i-1][j-1]+A[i-1][j]+A[i-1][j+1]) / 3;
```

Example

```
for (i = 0; i < N; ++i) {
  s[i] = i;
  for (j = 0; j < N; ++j)
    t[i] = s[i] + 1;
 }
```

Some Terminology on Dependence Relations

We categorize the dependence relation in three kinds:

- **Uniform dependences:** the distance between two dependent iterations is a constant
  - ex: $i \rightarrow i + 1$
  - ex: $i, j \rightarrow i + 1, j + 1$
- **Non-uniform dependences:** the distance between two dependent iterations varies along the execution
  - ex: $i \rightarrow i + j$
  - ex: $i \rightarrow 2i$
- **Parametric dependences:** at least a parameter is involved in the dependence relation
  - ex: $i \rightarrow i + N$
  - ex: $i + N \rightarrow j + M$

Data Dependence Analysis

Objective: compute the set of statement instances which are dependent

Some of the several possible approaches:

- **Distance vector:** compute an indicator of the distance between two dependent iteration
  - Problems: approximative for non-uniform dependences
- Best solution: **Dependence polyhedron**, list of sets of dependent instances, with a set of dependence polyhedra for each pair of statements

Dependence Polyhedron

Principle: model all **pairs** of instances in dependence

Definition (Dependence of statement instances)

A statement R depends on a statement S (written $S \rightarrow R$) if there exists an operation $S(\vec{x}_S)$ and $R(\vec{x}_R)$ and a memory location m such that:

1. $S(\vec{x}_S)$ and $R(\vec{x}_R)$ refer to the same memory location m, and at least one of them writes to that location,
2. x_S and x_R belongs to the iteration domain of R and S,
3. in the original sequential order, $S(\vec{x}_S)$ is executed before $R(\vec{x}_R)$.

Dependence Polyhedron

Data Dependent Graph

There is an edge between statements if they are depedent. And for each edge we define a dependence polyhedron for $S \rightarrow R$ at a given level l and for a given pair of references f_R, f_S

A few properties:

- We can always build the dep polyhedron for a given pair of affine array accesses (it is convex)
- It is exact, if the iteration domain and the access functions are also exact
- Can be represented as matrix, since all the constraints that define are affine

Legal Program Transformation

- A Transformation is illegal if a dependence crosses the hyperplane backwards
- A dependence going in forward direction between two hyperplanes indicates sequentiality
- No dependence between any point of the hyperplane indicates parallelism

Definition (Precedence condition)

Given $\mathbf{\Theta^T}$ a schedule for the instance of T, $\mathbf{\Theta^S}$ a schedule for the instance of S. $\mathbf{\Theta^T}$ and $\mathbf{\Theta^S}$ are legal schedules if
$\forall < \vec{x}_S, \vec{x}_T > \in D_{S,T} : \mathbf{\Theta^S} << \mathbf{\Theta^T}$
Equivalently: $\Delta_{T,S} = \mathbf{\Theta^T} - \mathbf{\Theta^S} >= 1$

Connection with Parallelism

- A dependence is loop-carried if 2 iterations of this loop access the same array cell
- If no such dependence exists, then loop is parallel

Example

```
for (i = 0; i < N; ++i)
  for (j = 0; j < N; ++j)
    C[i][j] = 0;
```

Example

```
for (i = 0; i < N; ++i)
  for (j = 0; j < N; ++j)
    A[i][j] = A[i][j -1]*B[i][j];
```

Polyhedral Representation of Programs

Static Control Parts

- Loops have affine control only (over-approximation otherwise)

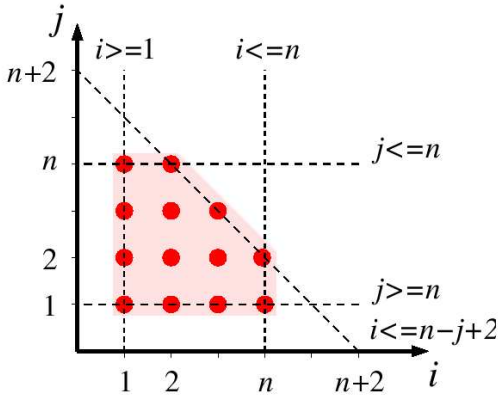
Polyhedral Representation of Programs

Static Control Parts

- Loops have affine control only (over-approximation otherwise)
- Iteration domain: represented as integer polyhedra

```
for (i=1; i<=n; ++i)
. for (j=1; j<=n; ++j)
. . if (i<=n-j+2)
. . . s[i] = ...
```

$$\mathcal{D}_{S1} = \begin{bmatrix} 1 & 0 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ -1 & -1 & 1 & 2 \end{bmatrix} \cdot \begin{pmatrix} i \\ j \\ n \\ 1 \end{pmatrix}$$

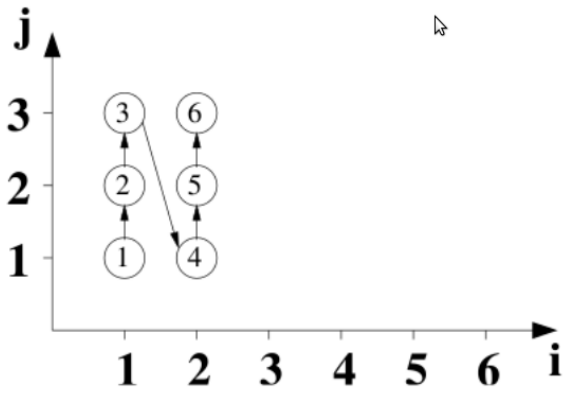


Iteration domain of S_1

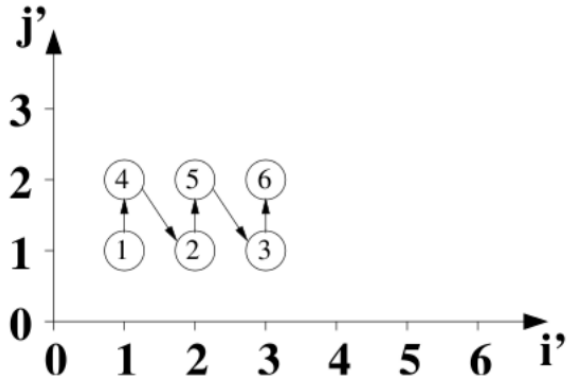
Polyhedral Representation of Programs

Static Control Parts

- Loops have affine control only (over-approximation otherwise)
- Iteration domain: represented as integer polyhedra
- Affine schedule: represented as Affine function



$\Longrightarrow$



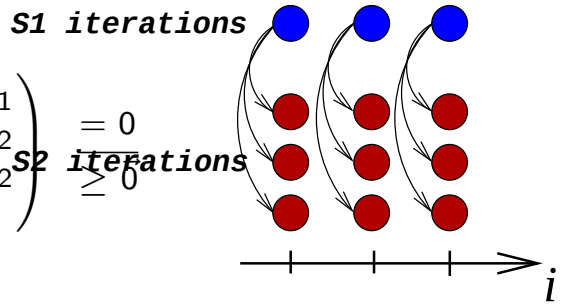
Polyhedral Representation of Programs

Static Control Parts

- Loops have affine control only (over-approximation otherwise)
- Iteration domain: represented as integer polyhedra
- Affine schedule: represented as Affine function
- Data dependence between S1 and S2: a subset of the Cartesian product of $\mathcal{D}_{S1}$ and $\mathcal{D}_{S2}$ (**exact analysis**)

```
for (i=1; i<=3; ++i) {
. s[i] = 0;
. for (j=1; j<=3; ++j)
. . s[i] = s[i] + 1;
}
```

$$\mathcal{D}_{S1\delta S2} : \left[\begin{array}{cccc} 1 & -1 & 0 & 0 \\ \hline 1 & 0 & 0 & -1 \\ -1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & -1 & 0 & 3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 3 \end{array}\right] \cdot \begin{pmatrix} i_{S1} \\ i_{S2} \\ j_{S2} \\ 1 \end{pmatrix} \begin{array}{c} = 0 \\ \hline \geq 0 \end{array}$$



Polyhedral Representation of Programs

Static Control Parts

- Loops have affine control only (over-approximation otherwise)
- Iteration domain: represented as integer polyhedra
- Affine schedule: represented as Affine function
- Data dependence between S1 and S2: a subset of the Cartesian product of $\mathcal{D}_{S1}$ and $\mathcal{D}_{S2}$ (**exact analysis**)
- Memory accesses: static references, represented as affine functions of $\vec{x_S}$ and $\vec{p}$

```
for (i=0; i<n; ++i) {
. s[i] = 0;
. for (j=0; j<n; ++j)
. . s[i] = s[i]+a[i][j]*x[j];
}
```

$$f_{\mathtt{s}}(\vec{x_{S2}}) = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} . \begin{pmatrix} \vec{x_{S2}} \\ n \\ 1 \end{pmatrix}$$

$$f_{\mathtt{a}}(\vec{x_{S2}}) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} . \begin{pmatrix} \vec{x_{S2}} \\ n \\ 1 \end{pmatrix}$$

$$f_{\mathtt{x}}(\vec{x_{S2}}) = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix} . \begin{pmatrix} \vec{x_{S2}} \\ n \\ 1 \end{pmatrix}$$

Program Transformations Example

Representing a schedule

```
 for (i = 0; i < n; ++i)
  for (j = 0; j < n; ++j){
S1: C[i][j] = 0;
   for (k = 0; k < n; ++k)
S2:  C[i][j] += A[i][k]*
                B[k][j];
  }
```

$$\Theta^{S1}.\vec{x}_{S1} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} i \\ j \\ n \\ 1 \end{pmatrix}$$

$$\Theta^{S2}.\vec{x}_{S2} = \begin{pmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} i \\ j \\ k \\ n \\ 1 \end{pmatrix}$$

```
for (i = n; i < 2*n; ++i)
 for (j = 0; j < n; ++j)
  C[i][j] = 0;
for (k= n+1; k<= 2*n; ++k)
 for (j = 0; j < n; ++j)
  for (i = 0; i < n; ++i)
   C[i][j] += A[i][k-n-1]*
              B[k-n-1][j];
```

	Transformation	Description
$\vec{\imath}$	`reversal`	Changes the direction in which a loop traverses its iteration range
	`skewing`	Makes the bounds of a given loop depend on an outer loop counter
	`interchange`	Exchanges two loops in a perfectly nested loop, a.k.a. `permutation`
$\vec{p}$	`fusion`	Fuses two loops, a.k.a. `jamming`
	`distribution`	Splits a single loop nest into many, a.k.a. `fission` or `splitting`
c	`peeling`	Extracts one iteration of a given loop
	`shifting`	Allows to reorder loops

Program Transformations Example

Original Schedule

```
 for (i = 0; i < n; ++i)
  for (j = 0; j < n; ++j){
S1: C[i][j] = 0;
   for (k = 0; k < n; ++k)
S2:  C[i][j] += A[i][k]*
                B[k][j];

  }
```

$$\Theta^{S1}.\vec{x}_{S1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} \mathbf{i} \\ \mathbf{j} \\ \mathbf{n} \\ \mathbf{1} \end{pmatrix}$$

$$\Theta^{S2}.\vec{x}_{S2} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} \mathbf{i} \\ \mathbf{j} \\ \mathbf{k} \\ \mathbf{n} \\ \mathbf{1} \end{pmatrix}$$

```
for (i = 0; i < n; ++i)
 for (j = 0; j < n; ++j){
  C[i][j] = 0;
  for (k = 0; k < n; ++k)
   C[i][j] += A[i][k]*
              B[k][j];

 }
```

Program Transformations Example

Distribute loops

```
 for (i = 0; i < n; ++i)
  for (j = 0; j < n; ++j){
S1: C[i][j] = 0;
   for (k = 0; k < n; ++k)
S2:  C[i][j] += A[i][k]*
                B[k][j];
  }
```

$$\Theta^{S1}.\vec{x}_{S1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} \mathbf{i} \\ \mathbf{j} \\ \mathbf{n} \\ \mathbf{1} \end{pmatrix}$$

$$\Theta^{S2}.\vec{x}_{S2} = \begin{pmatrix} 1 & 0 & 0 & \mathbf{1} & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} \mathbf{i} \\ \mathbf{j} \\ \mathbf{k} \\ \mathbf{n} \\ \mathbf{1} \end{pmatrix}$$

```
for (i = 0; i < n; ++i)
 for (j = 0; j < n; ++j)
  C[i][j] = 0;
for (i = n; i < 2*n; ++i)
 for (j = 0; j < n; ++j)
  for (k = 0; k < n; ++k)
   C[i-n][j] += A[i-n][k]*
                B[k][j];
```

Program Transformations Example

Distribute loops + Interchange loops for S2

```
 for (i = 0; i < n; ++i)
  for (j = 0; j < n; ++j){
S1: C[i][j] = 0;
   for (k = 0; k < n; ++k)
S2:  C[i][j] += A[i][k]*
                B[k][j];

  }
```

$$\Theta^{S1}.\vec{x}_{S1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} . \begin{pmatrix} \mathbf{i} \\ \mathbf{j} \\ \mathbf{n} \\ \mathbf{1} \end{pmatrix}$$

$$\Theta^{S2}.\vec{x}_{S2} = \begin{pmatrix} 0 & 0 & \mathbf{1} & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \mathbf{1} & 0 & 0 & 0 & 0 \end{pmatrix} . \begin{pmatrix} \mathbf{i} \\ \mathbf{j} \\ \mathbf{k} \\ \mathbf{n} \\ \mathbf{1} \end{pmatrix}$$

```
for (i = 0; i < n; ++i)
 for (j = 0; j < n; ++j)
  C[i][j] = 0;
for (k = n; k < 2*n; ++k)
 for (j = 0; j < n; ++j)
  for (i = 0; i < n; ++i)
   C[i][j] += A[i][k-n]*
              B[k-n][j];
```

Program Transformations Example

Illegal schedule

```
 for (i = 0; i < n; ++i)
  for (j = 0; j < n; ++j){
S1: C[i][j] = 0;
   for (k = 0; k < n; ++k)
S2:  C[i][j] += A[i][k]*
                B[k][j];

  }
```

$$\Theta^{S1}.\vec{x}_{S1} = \begin{pmatrix} 1 & 0 & \mathbf{1} & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} \mathbf{i} \\ \mathbf{j} \\ \mathbf{n} \\ \mathbf{1} \end{pmatrix}$$

$$\Theta^{S2}.\vec{x}_{S2} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} \mathbf{i} \\ \mathbf{j} \\ \mathbf{k} \\ \mathbf{n} \\ \mathbf{1} \end{pmatrix}$$

```
for (k = 0; k < n; ++k)
 for (j = 0; j < n; ++j)
  for (i = 0; i < n; ++i)
   C[i][j] += A[i][k]*
              B[k][j];
for (i = n; i < 2*n; ++i)
 for (j = 0; j < n; ++j)
  C[i-n][j] = 0;
```

Program Transformations Example

Implicit fine-grain parallelism

```
 for (i = 0; i < n; ++i)
  for (j = 0; j < n; ++j){
S1: C[i][j] = 0;
   for (k = 0; k < n; ++k)
S2:  C[i][j] += A[i][k]*
                B[k][j];
  }
```

$$\Theta^{S1}.\vec{x}_{S1} = \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix} . \begin{pmatrix} \mathbf{i} \\ \mathbf{j} \\ \mathbf{n} \\ \mathbf{1} \end{pmatrix}$$

$$\Theta^{S2}.\vec{x}_{S2} = \begin{pmatrix} 0 & 0 & 1 & 1 & 0 \end{pmatrix} . \begin{pmatrix} \mathbf{i} \\ \mathbf{j} \\ \mathbf{k} \\ \mathbf{n} \\ \mathbf{1} \end{pmatrix}$$

```
for (i = 0; i < n; ++i)
 pfor (j = 0; j < n; ++j)
  C[i][j] = 0;
for (k = n; k < 2*n; ++k)
 pfor (j = 0; j < n; ++j)
  pfor (i = 0; i < n; ++i)
   C[i][j] += A[i][k-n]*
              B[k-n][j];
```

References

- **http://polly.llvm.org/**
- Chapter 11: Optimizing for Parallelism and Locality from **Compiler Principles and Techniques and Tools** book
- http://pluto-compiler.sourceforge.net/
- http://www.cse.ohio-state.edu/ pouchet/software/pocc/

Thank you
Any Questions?
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